

A FIFTH-ORDER BI-HAMILTONIAN SYSTEM

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ABSTRACT. In this work, we introduced a new two component fifth-order bi-Hamiltonian system admitting the scalar Kupershmidt equation as a reduction.

INTRODUCTION

To prove the integrability of an equation suspected to be bi-Hamiltonian, one need to find an appropriate compatible pair of Hamilton operator J and K such that the Magri scheme

$$u_{t_i} = F_i[u] = KG_i[u] = JG_{i+1}[u], \quad i = -1, 0, 1, 2, 3, \dots$$

constructed by the operators contains the equation in hand. Here $F_i[u]$ are characteristics of symmetries and G_i are the conserved gradients.

The scalar equations of order up to 5 are extensively classified with respect to existence of sufficiently many higher conserved densities for the existence of a formal symmetry. Unlike lower order cases, classification of fifth order two-component evolution equations is an obstinate problem. Some completely integrable equations of this type were found by Mikhailov, Novikov and Wang [6, 5] in the study of symbolic representation theory and nonevolutionary equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -\frac{5}{3}u_{5x} - 10vv_{3x} + 10uu_{3x} + 25u_xu_{xx} - 15v_xv_{xx} - 12u^2u_x \\ \quad + 6v^2u_x + 12uvv_x - 6v^2v_x \\ 15v_{5x} - 10vv_{3x} - 30uv_{3x} - 35v_xu_{xx} + 30v_xv_{xx} - 45u_xv_{xx} \\ \quad + 6v^2u_x - 6v^2v_x + 12uvu_x + 12u^2v_x \end{pmatrix}, \quad (1)$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{5x} + 10uu_{3x} + 25u_xu_{xx} + 20u^2u_x + v^2v_x \\ u_{3x}v + u_{xx}v_x + 8uvu_x + 4u^2v_x \end{pmatrix}, \quad (2)$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{8}u_{5x} - 2uu_{3x} - 2u_xu_{xx} - \frac{32}{5}u^2u_x + v_x \\ \frac{9}{8}v_{5x} + 6uv_{3x} + 6u_xv_{xx} + 4u_{xx}v_x + \frac{32}{5}u^2v_x \end{pmatrix}. \quad (3)$$

System (1) and (2) admit a reduction $v = 0$ to the Kaup-Kupershmidt equation $u_t = u_{5x} + 10uu_{3x} + 25u_xu_{xx} + 20u^2u_x$ and By setting $v = 0$, system (3) reduces to the Sawada-Kotera equation $u_t = u_{5x} + 5uu_{3x} + 5u_xu_{xx} + 5u^2u_x$ (see [1, 3] and references therein).

Bi-Hamiltonian structures for (2) and (3) can be found in [5]. Bi-Hamiltonian structure for system (1) and Zero curvature representation for (2) are discussed in [7, 4]. Very recently system (2) considered by De Sole, Kac and Turhan [8] in the study of the Lenard-Magri scheme of integrability who developed a new method based on the notion of strongly skew-adjoint differential operators, using the Lie superalgebra of variational polyvector fields. However, no two-component completely integrable system with reduction $v = 0$ to the kupershmidt equation is known so far. In this work, we introduce a new system of this type whose bi-Hamiltonian structure we constructed too.

THE NEW SYSTEM

The new fifth order bi-Hamiltonian two-component system we introduce here is

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{5x} - 30vv_{4x} + 5u_xu_{3x} - 5u^2u_{3x} + 15v^2u_{3x} - 75v_xv_{3x} + 60uvv_{3x} \\ + 90v^2v_{3x} + 5u_{xx}^2 - 20uu_xu_{xx} + 60vv_xu_{xx} - 45v_{xx}^2 + 90vu_xv_{xx} \\ + 90uv_xv_{xx} + 540vv_xv_{xx} + 30u^2vv_{xx} - 180uv^2v_{xx} - 90v^3v_{xx} \\ - 5u_x^3 + 45u_xv_x^2 + 60uvv_xv_x - 180v^2u_xv_x + 5u^4u_x \\ - 90u^2v^2u_x + 45v^4u_x + 180v_x^3 + 30u^2v_x^2 - 360uvv_x^2 \\ - 270v^2v_x^2 - 60u^3vv_x + 180uv^3v_x \\ - 9v_{5x} + 10vu_{4x} + 25v_xu_{3x} + 20uvv_{3x} + 30v^2u_{3x} + 15u_xv_{3x} + 90v_xv_{3x} \\ + 15u^2v_{3x} + 15v^2v_{3x} + 30u_{xx}v_{xx} + 50vv_xu_{xx} - 10u^2vu_{xx} + 50uv_xu_{xx} \\ + 60vv_xu_{xx} + 60uv^2u_{xx} + 30v^3u_{xx} + 90v_{xx}^2 + 60uu_xv_{xx} + 60vv_xv_{xx} \\ + 45u_x^2v_x - 20uvv_x^2 + 60v^2u_x^2 - 10u^2u_xv_x + 90v^2u_xv_x \\ - 20u^3vu_x + 120uvv_xv_x + 60uv^3u_x + 15v_x^3 - 5u^4v_x \\ + 90u^2v^2v_x - 45v^4v_x. \end{pmatrix}. \quad (4)$$

By setting $v = 0$ the well known Kupershmidt equation is an obvious reduction of system (4):

$$u_t = u_{5x} + 5u_xu_{3x} + 5u_{xx}^2 - 5u^2u_{3x} - 20uu_xu_{xx} - 5u_x^3 + 5u^4u_x.$$

Proposition 1. *System (4) can be written in Hamiltonian form in not just one but two different ways:*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = F_1[u, v] = J \begin{pmatrix} \delta_u \\ \delta_v \end{pmatrix} \int \rho_1 \, dx = K \begin{pmatrix} \delta_u \\ \delta_v \end{pmatrix} \int \rho_{-1} \, dx \quad (5)$$

with the compatible pair of Hamiltonian operators

$$J = \begin{pmatrix} 3D_x & 0 \\ 0 & D_x \end{pmatrix}, K = \begin{pmatrix} K_1 & K_2 \\ -K_2^* & K_4 \end{pmatrix}$$

where

$$K_1 = 2D_x^7 + \alpha_1 D_x^5 + D_x^5 \alpha_1 + \alpha_2 D_x^3 + D_x^3 \alpha_2 + \alpha_3 D_x + D_x \alpha_3 + 4u_x D_x^{-1} u_t + 4u_t D_x^{-1} u_x$$

$$K_2 = -56D_x^6 v + D_x^5 \alpha_4 + D_x^4 \alpha_5 + D_x^3 \alpha_6 + D_x^2 \alpha_7 + D_x \alpha_8 + \alpha_9 + 4u_x D_x^{-1} v_t + 4v_t D_x^{-1} v_x$$

$$K_4 = -18D_x^7 + \alpha_{10} D_x^5 + D_x^5 \alpha_{10} + \alpha_{11} D_x^3 + D_x^3 \alpha_{11} + \alpha_{12} D_x + D_x \alpha_{12} + 4v_x D_x^{-1} v_t + 4v_t D_x^{-1} v_x$$

where

$$\alpha_1 = 6(u_x - u^2 + 11v^2)$$

$$\alpha_2 = -16u_{3x} + 20uu_{xx} - 336vv_{xx} + 29u_x^2 - 6u^2u_x - 381v_x^2 + 9u^4 - 294u^2v^2 + 264uvv_x + 45v^4 + 234v^2u_x - 180v^2v_x$$

$$\alpha_3 = 2(5u_{5x} + 102vv_{4x} - 25u_xu_{3x} + 3u^2u_{3x} - 117v^2u_{3x} + 453v_xv_{3x} - 72uvv_{3x} + 90v^2v_{3x} - 21u_{xx}^2 - 600vv_xu_{xx} + 8uu_xu_{xx} - 8u^3u_{xx} + 264uv^2u_{xx} + 351v_{xx}^2 - 498vu_xv_{xx} + 540vv_xv_{xx} - 306uvv_xv_{xx} + 174u^2vv_{xx} - 180uv^2v_{xx} - 90v^3v_{xx} + 6u_x^3 - 498u_xv_x^2 + 180v_x^3 + 264u^2v_x^2 - 360uvv_x^2 - 270v^2v_x^2 - 2u^6 + 60u^4v^2 - 60u^3vv_x - 44u^2u_x^2 - 90u^2v^4 + 360u^2v^2v_x - 6uu_{4x} + 180uv^3v_x + 1116uvv_xv_x + 294v^2u_x^2)$$

$$\alpha_4 = 4(53v_x + 28uv + 42v^2)$$

$$\alpha_5 = 2(-165v_{xx} - 192vu_x - 212uv_x - 312vv_x + 32u^2v - 168uv^2 - 72v^3)$$

$$\alpha_6 = 4(66v_{3x} + 141vu_{xx} + 165uv_{xx} + 90vv_{xx} + 281u_xv_x + 45v_x^2 - 32u^3v - 37u^2v_x - 6u^2v^2 + 72uv^3 - 86uvu_x + 312uvv_x - 18v^4 + 246v^2u_x + 177v^2v_x)$$

$$\alpha_7 = 2(-54v_{4x} - 194vu_{3x} - 264uv_{3x} - 36vv_{3x} - 557v_xu_{xx} + 168uvu_{xx} - 462v^2u_{xx} - 603u_xv_{xx} + 162v_xv_{xx} + 57u^2v_{xx} - 360uvv_{xx} - 63v^2v_{xx} - 288v^3u_x + 184vv_x^2 - 1224vu_xv_x + 108uv^2u_x + 306uu_xv_x - 180uv_x^2 - 246vv_x^2 + 148u^3v_x + 24u^3v^2 + 336u^2vu_x + 24u^2vv_x - 708uv^2v_x - 36v^3v_x - 4u^4v + 72uv^4 + 36v^5)$$

$$\alpha_8 = 2(9v_{5x} + 46vu_{4x} + 189v_xu_{3x} - 80uvu_{3x} + 138v^2u_{3x} + 249u_xv_{3x} - 90v_xv_{3x} - 15u^2v_{3x} + 72uvv_{3x} - 15v^2v_{3x} + 300u_{xx}v_{xx} - 90v_{xx}^2 - 174uu_xv_{xx} + 360vu_xv_{xx} - 120vv_xv_{xx} - 114u^3v_{xx} - 324uvv_xv_{xx} + 126uv^2v_{xx} - 190u_x^2v_x + 180u_xv_x^2 - 30v_x^3 + 8u^5v + 4u^4v_x + 36u^3vv_x - 48u^3vv_x - 456u^2u_xv_x - 192u^2v^2u_x - 60u^2v^2v_x - 174u^2vu_{xx} + 108uv_{4x} - 204uv_xu_{xx} - 72uv^5 - 60uv^3u_x + 72uv^3v_x - 144uv^2u_{xx} - 308uvu_x^2 - 168uvv_xv_x + 492uvv_x^2 - 72v^4u_x + 114v^3u_{xx} - 84v^2u_x^2 + 648v^2u_xv_x + 180v^2v_x^2 - 222vu_xu_{xx} + 564vv_xu_{xx})$$

$$\alpha_9 = 4(-9uv_{5x} + v_xu_{4x} + 10uvu_{4x} + 20u^2vu_{3x} + 25uvv_{3x} + 30uv^2v_{3x} + 15u^3v_{3x} + 15uu_xv_{3x} + 90uvv_{3x} + 15uv^2v_{3x} - 30vv_xv_{3x} + 5u_xv_xu_{xx} + 30uu_{xx}v_{xx} + 90uv_{xx}^2 - 45v_x^2v_{xx} + 60u^2u_xv_{xx} + 120uvv_xv_{xx} + 90v^2v_xv_{xx} + 30u^2vv_x^2 + 30vu_xv_x^2 + 30uv_x^3 - 4u^5v_x - 20u^4vv_x - 10u^3u_xv_x + 60u^3v^2v_x - 10u^3vu_{xx} + 45u^2v_xu_{xx} + 60u^2v^3u_x + 60u^2v^2u_{xx} - 20u^2vv_x^2 + 120u^2vu_xv_x + 40uu_x^2v_x + 30uv^3u_{xx} + 60uv^2u_x^2 + 90uv^2u_xv_x - 180uv^2v_x^2 + 50uvu_xu_{xx} + 60uvv_xu_{xx} - 90v^3v_x^2 + 15v^2v_xu_{xx} + 180vv_x^3)$$

$$\alpha_{10} = 2(9u_x + 54v_x + 9u^2 + 13v^2)$$

$$\alpha_{11} = -36u_{3x} - 288v_{3x} - 72uu_{xx} + 72vu_{xx} - 68vv_{xx} - 81u_x^2 - 36u_xv_x - 18u^2u_x + 144uvu_x + 62v^2u_x - 275v_x^2 - 36u^2v_x - 192v^2v_x - 9u^4 + 62u^2v^2 - 69v^4$$

$$\alpha_{12} = 2(9u_{5x} + 90v_{5x} + 18uu_{4x} - 36vu_{4x} + 26vv_{4x} + 81u_xu_{3x} - 54v_xu_{3x} + 9u^2u_{3x} - 72uvu_{3x} - 51v^2u_{3x} + 18u_xv_{3x} + 239v_xv_{3x} + 18u^2v_{3x} + 96v^2v_{3x} + 63u_{xx}^2 + 54uu_xu_{xx} - 216vu_xu_{xx} - 108uv_xu_{xx} - 154vv_xu_{xx} + 18u^3u_{xx} - 102uv^2u_{xx} - 60v^3u_{xx} + 231v_{xx}^2 - 92vu_xv_{xx} + 396vv_xv_{xx} - 92u^2vv_{xx} + 108v^3v_{xx} + 18u_x^3 - 92v^2u_x^2 - 108u_x^2v_x + 54u^2u_x^2 - 62u_xv_x^2 + 192v_x^3 + 20u^2v^2u_x - 120uv^3u_x - 308uvv_xv_x - 60v^4u_x + 384v^2v_x^2 + 18v^6 + 10u^4v^2 - 62u^2v_x^2 - 60u^2v^4)$$

By straightforward calculation it is easy to show that the functional trivector of linear combination $K + \lambda J$ with constant λ , vanishes independently from the value of λ [2].

The first few conserved densities of the hierarchy are listed below.

$$\begin{aligned}
\rho_{-1} &= \alpha \\
\rho_0 &= u^2 + 3v^2 \\
\rho_1 &= 3uu_{4x} - 81vv_{4x} - 5u^2u_{3x} + 45v^2u_{3x} - 90uvv_{3x} - 270v^2v_{3x} - 5u^3u_{xx} + 45uv^2u_{xx} + 180v^3u_{xx} \\
&\quad + 135u^2vv_{xx} + 45v^3v_{xx} + 60u^3vv_x - 540u^2v^2v_x - 540uv^3v_x + u^6 \\
&\quad - 45u^4v^2 + 135u^2v^4 - 27v^6 \\
\rho_2 &= -90uu_{6x} + 7290vv_{6x} + 126u^2u_{5x} - 756v^2u_{5x} + 13608uvv_{5x} + 20412v^2v_{5x} + 105u^3u_{4x} \\
&\quad - 5670uv^2u_{4x} - 3780v^3u_{4x} + 11340uv_xv_{4x} - 11340u^2vv_{4x} - 34020uv^2v_{4x} + 16065v^3v_{4x} \\
&\quad - 15120uvv_xu_{3x} - 70u^4u_{3x} + 10080u^2v^2u_{3x} - 7560uv^3u_{3x} - 2835v^4u_{3x} - 136080uvv_xv_{3x} \\
&\quad - 7140u^3vv_{3x} + 45360u^2v^2v_{3x} + 11340uv^3v_{3x} - 17010v^4v_{3x} + 315u^2u_{xx}^2 - 11340uvu_{xx}v_{xx} \\
&\quad + 18900u^2vv_xu_{xx} - 22680uv^2v_xu_{xx} - 252u^5u_{xx} + 8820u^3v^2u_{xx} + 13608v^5u_{xx} - 5670u^2v^2v_{xx}^2 \\
&\quad - 68040uvv_{xx}^2 - 76545v^2v_{xx}^2 - 6300u^3v_xv_{xx} + 136080u^2vv_xv_{xx} + 34020uv^2v_xv_{xx} \\
&\quad + 6930u^4vv_{xx} + 13608v^5v_{xx} + 7560u^3v^2v_{xx} - 22680u^2v^3v_{xx} - 4410u^4v_x^2 + 15120u^3vv_x^2 \\
&\quad - 34020u^2v^2v_x^2 + 1512u^5vv_x - 11340u^4v^2v_x - 68040u^2v^4v_x - 68040uv^5v_x + 30u^8 \\
&\quad - 1260u^6v^2 + 11340u^2v^6 - 2430v^8 \\
&\vdots \\
&\vdots \\
&\vdots
\end{aligned}$$

These densities suffice to write two Magri schemes with same hamiltonian operators that one of them contains the new system proving integrability of the system (4).

The recursion relation

$$F_n = J \begin{pmatrix} \delta_u \\ \delta_v \end{pmatrix} \int \rho_n \, dx = K \begin{pmatrix} \delta_u \\ \delta_v \end{pmatrix} \int \rho_{n-2} \, dx \quad (6)$$

is a more general relation of (5) that will provide further conservation laws for n-th stage symmetry of the original system.

let us also mention that the recursion operator can be written in the form $R = KJ^{-1}$. Starting with the basic symmetries $\begin{pmatrix} u_x \\ v_x \end{pmatrix}$ and $\begin{pmatrix} u_t \\ v_t \end{pmatrix}$ we can generate two infinite generalized symmetries by applying

$$K_{n+2} = RK_n. \quad (7)$$

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